ON A PARTICULAR CASE OF DAMPING OF A GYROSTAT ROTATION

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Studies of various stabilization methods of a rigid body with respect to specially oriented axes are at present of great interest [1 and 4].

Among them the method which uses the inertial pendulum masses possesses a number of important technical advantages [4].

It is of interest to consider the application of this method to the problem of constructing an optimum law of stabilization of rotations of a rigid body about a fixed point [5 and 8].

1. Statement of the problem. Let us consider a free rigid body fixed at the center of its mass. Let $OX_1X_2X_3$ be the stationary coordinate system, $Ox_1x_2x_3$ be the coordinate system attached to the body and oriented along the principal central axes of inertia (Fig.1.).

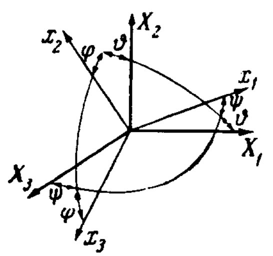

Fig. 1

The orientation of the system $Ox_1x_2x_3$ with respect to $OX_1X_2X_3$ can be determined for example by the Eulerian angles. We shall assume that our body contains flywheels whose axes coincide with the $X_1X_2X_3$ axes.

By definition such a system is called a gyrostat. The equations of motion of a gyrostat from the principle of angular momentum were given by Volterra [9].

They have the form

$$C_1\dot{p_1} + J_1\dot{\omega_1} + (C_3 - C_2)p_2p_3 + H_3p_2 - H_2p_3 = 0 \quad (1\ 2\ 3)$$
$$H_i = J_i\omega_i \qquad (i = 1, 2, 3) \qquad (1.1)$$

Here C_1, C_2, C_3 are the principal central moments of inertia of the gyrostat (assuming that the flywheels do not rotate); J_1, J_2, J_3 are moments of inertia of the flywheels; ω_1, ω_2, ω_3 are the angular velocities of the flywheels with respect to the body; p_1, p_2, p_3 are the x_1, x_2, x_3 components of the angular velocity p of the body.

The symbol (1, 2, 3) means that the two following equations are obtained from (1.1) by cyclic permutations.

The equations of motion of flywheels are

$$J_i(\omega_i^{\cdot} + p_i^{\cdot}) = -U_i \qquad (i = 1, 2, 3) \qquad (1\ 2\ 3) \qquad (1.2)$$

Here U_i are moments of the motors which drive the flywheels. Let

$$A_i = C_i - J_i \qquad (i = 1, 2, 3)$$

From Equations(1.1) and (1.2) we find

$$A_1 p_1^{\cdot} = U_1 + (C_2 p_2 + H_2) p_3 - (C_3 p_3 + H_3) p_2 \qquad (1\,2\,3) \tag{1.3}$$

Equations (1.3) are the ones which we shall investigate. Here U_1 are considered as moments controlling the body.

Let us turn to Equations (1.2) and (1.3). We shall introduce the new variables

$$z_i = A_i p_i + J_i (\omega_i + p_i) \qquad (i = 1, 2, 3) \tag{1.4}$$

Then the combined equations will take the form

$$p_1^{\cdot} = A^{-1} (z_2 p_3 - z_3 p_2 + U_1) \qquad (1\,2\,3) \tag{1.5}$$

$$z_1^{\cdot} = z_2 p_3 - z_3 p_2 \qquad (1\,2\,3) \tag{1.6}$$

Equations (1.6) permit the first integral

$$z_1^2 + z_2^2 + z_3^2 = C, \qquad C = \text{const} \tag{1.7}$$

It follows that if $z_1(0) + z_2(0) + z_3(0) < \infty$, then the functions z_i $(i = 1, 2, 3)$ are always bounded.

Existence of the integral shows, that in our case it is impossible, in general, to reduce to zero all the variables p_i, ω_i $(i = 1, 2, 3)$. It would be possible only in a very particular case when the motion takes place in the subspace of the initial states

$$z_{i0} = 0 \qquad (i = 1, 2, 3) \tag{1.8}$$

Consequently, the system (1.5), (1.6) cannot be stabilized with respect to all the variables p_i, ω_i $(i = 1, 2, 3)$. This property does not negate the conditions of stability of the system described in [10].

Thus, we shall attempt to investigate the problem of stabilization of rotation of the rigid body alone, with any arbitrary functions z_i $(i = 1, 2, 3)$ being bounded and satisfying the conditions (1.6), (1.7).

We shall stabilize the rotations of a rigid body about a fixed point by means of flywheels. That such stabilization is possible follows from the angular momentum theorem, which says that every directional motion of the flywheels causes a directional motion of the rigid body.

Every flywheel is driven by an electric motor, consequently, the control of the body is reduced to the control of the voltage transmitted to the driving motors.

If $U_i = 0$ $(i = 1, 2, 3)$, then Equations (1.3) have an obvious solution

$$p_1^* = p_2^* = p_3^* = 0 \tag{1.9}$$

which must be stabilized.

As to the solution (1.9), Equations (1.5) can be treated as equations of a perturbed motion, valid when the arbitrary p_i $(i = 1, 2, 3)$ are bounded.

Equations (1.5) shall be regarded as the initial equations of the controlled object which are valid at any z_i $(i = 1, 2, 3)$ satisfying the condition (1.6), (1.7).

We shall treat Equations (1.5) as equations with variable coefficients. We shall assume that:

a) the equations of a perturbed motion (1.5) of the controlled object are given,

b) the equations (1.5) determine a multitude of perturbed motions of the object occurring in a certain neighborhood

$$p_{10}^2 + p_{20}^2 + p_{30}^2 \leqslant A, \qquad A = \text{const} > 0 \tag{1.10}$$

of the state (1.9)

c) the optimizing functional is

$$I = \int_0^\infty W dt \qquad (1.11)$$

in which W is a sign-definite positive function of the form

$$W = \sum_{i=1}^{3} (a p_i^2 + U_i^2) \qquad (a > 0) \qquad (1.12)$$

of the variables p_i, U_i, where a is a weight constant.

Every control U_i of piecewise continuous class at which $I < \infty$ will be called permissible.

P r o b l e m . Find control

$$U_i = U_i(p, t) \qquad (i = 1, 2, 3) \qquad (1.13)$$

at which Equations (1.5) are satisfied (at any z_i ($i = 1, 2, 3$) satisfying conditions (1.6), (1.7), and at which the functional (1.11) would have a minimum for all motions occurring in the neighborhood of (1.10).

2. Solution of the problem. In order to solve the problem we shall use methods of dynamic programing. We shall introduce the following notation

$$\Psi(p(t), t) = \min_U \int_t^\infty W dt, \qquad U = \{u_1, u_2, u_3\} \qquad (2.1)$$

Let us write down the equation of Bellman. It can be expressed in the following form

$$0 = \min_U \left\{ a \sum_{i=1}^{3} p_i^2 + \frac{\partial \Psi}{\partial t} + A_1^{-1}(z_2 p_3 - z_3 p_2) \frac{\partial \Psi}{\partial p_1} + A_2^{-1}(z_3 p_1 - z_1 p_3) \frac{\partial \Psi}{\partial p_2} + \right.$$
$$\left. + A_3^{-1}(z_1 p_2 - z_2 p_1) \frac{\partial \Psi}{\partial p_3} + \sum_{i=1}^{3} \left(U_i + \frac{1}{2} A_i^{-1} \frac{\partial \Psi}{\partial p_i} \right)^2 - \frac{1}{4} \sum_{i=1}^{3} \left(A_i^{-1} \frac{\partial \Psi}{\partial p_i} \right)^2 \right\} \qquad (2.2)$$

from which we find the optimum control

$$U_i = -\frac{1}{2} A_i^{-1} \frac{\partial \Psi}{\partial p_i} \qquad (i = 1, 2, 3) \qquad (2.3)$$

It is obvious that in order to solve the problem we must find first the function Ψ, which we call the generating function. It should satisfy the following equation

$$-\frac{\partial \Psi}{\partial t} = a \sum_{i=1}^{3} p_i^2 + A_1^{-1}(z_2 p_3 - z_3 p_2) \frac{\partial \Psi}{\partial p_1} + A_2^{-1}(z_3 p_1 - z_1 p_3) \frac{\partial \Psi}{\partial p_2} +$$
$$+ A_3^{-1}(z_1 p_2 - z_2 p_1) \frac{\partial \Psi}{\partial p_3} - \frac{1}{4} \sum_{i=1}^{3} \left(A_i^{-1} \frac{\partial \Psi}{\partial p_i} \right)^2 \qquad (2.4)$$

and also the condition

$$\Psi(p(\infty), \infty) = 0 \qquad (2.5)$$

The equation of Bellman can be satisfied if we set

$$\Psi = \rho \sum_{i=1}^{3} A_i p_i^2, \qquad \rho^2 = a \qquad (i = 1, 2, 3) \qquad (2.6)$$

It is seen that in this case the function Ψ which is the solution of our problem and expresses the control law, does not depend on t explicitly.

Consequently, the law of stabilization is determined from Formulas

$$U_i = -\rho p_i, \qquad \rho = \sqrt{a} \qquad (2.7)$$

The closed system has the form

$$A_1 p_1^{\cdot} = z_2 p_3 - z_3 p_2 - \rho p_1 \qquad (1\,2\,3) \tag{2.8}$$

3. Dynamics of a closed system. 1. Asymptotic stability. The state (1.9) is asymptotically stable at any z_i, satisfying Equations (1.6),(1.7). Indeed, the function Ψ is sign-definite, positive with respect to p_i (i = 1, 2, 3), and its total derivative along the trajectories of the system (2.8) is

$$\Psi^{\cdot} = -W \tag{3.1}$$

Consequently, the conditions of the second Liapunov theorem on asymptotic stability are satisfied.

On the strength of the asymptotic stability of the state (1.9) (at any z_i (i = 1, 2, 3), satisfying the conditions (1.6), (1.7)),the function Ψ satisfies the condition (2.5) and the integral (2.1) is bounded.

2. Rate of damping of Ψ. We have

$$\Psi^{\cdot} = -2a \sum_{i=1}^{3} p_i^2 \qquad (i = 1, 2, 3) \tag{3.2}$$

a) Let $A_1 = A_2 = A_3 = A$. Then we have the integral

$$\sum_{i=1}^{3} p_i^2 = \sum_{i=1}^{3} p_{i0}^2 \exp\left(-\frac{2\sqrt{a}}{A_1} t\right) \tag{3.3}$$

b) In the general case $A_1 < A_2 < A_3$, hence

$$\rho A_1 \sum_{i=1}^{3} p_{i0}^2 \exp\left(-\frac{2\sqrt{a}}{A_1} t\right) \leqslant \Psi \leqslant \rho A_3 \sum_{i=1}^{3} p_{i0}^2 \exp\left(-\frac{2\sqrt{a}}{A_3} t\right) \tag{3.4}$$

The inequality (3.4) permits to obtain two-sided estimates on the rate of damping for each velocity component of the rigid body.

3. The first integral. Equations (1.6) have the first integral

$$z_1^2 + z_2^2 + z_3^2 = \text{const} \tag{3.5}$$

Since $p_i(\infty) = 0$ ($i = 1, 2, 3$), we have

$$\left(\sum_{i=1}^{3} J_i \omega_i\right)_\infty = \text{const} \tag{3.6}$$

Existence of the limiting value of the integral confirms the conclusion that motions of flywheels are not controllable.

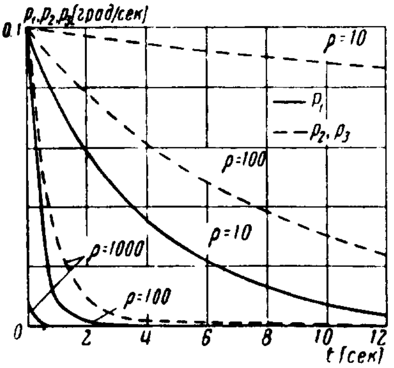

Fig. 2

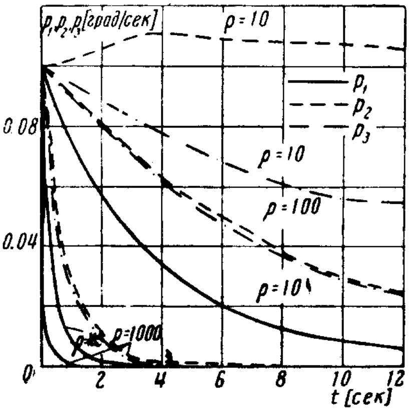

Fig. 3

4. Dynamics. Dynamics of a perturbed motion, and especially the rate of its damping can be investig-ted by integrating Equations (2.8) by the method of successive approximations.

The first approximation has the form

$$p_{i1} = p_{i0}e^{-\lambda_i t}, \qquad \omega_{i1} = \omega_{i0} + \frac{\lambda_i + \varphi_i}{\lambda_i} p_{i0}(1 - e^{-\lambda_i t}) \qquad (i = 1, 2, 3) \qquad (3.7)$$

All the following approximations are found from Formulas

$$p_{ij}^{\cdot} = -\lambda_i p_{ij} + f_{i,j-1}(t), \quad \omega_{ij}^{\cdot} = (\lambda_i + \varphi_i) p_{ij} - f_{ij-1}(t) \qquad (i = 1, 2, 3) \quad (3.8)$$

Here

$$\lambda_i = \frac{\rho}{A_i}, \; \varphi_i = \frac{\rho}{J_i}, \quad f_{1,j-1}(t) = (J_2 - J_3) p_{2,j-1} p_{3,j-1} + J_2\omega_{2,j-1} p_{3,j-1} - J_3\omega_{3,j-1} p_{2,j-1} \quad (1\ 2\ 3) \qquad (3.9)$$

Agreement of these approximations can be proved. In a number of concrete cases we can use only the first two approximations.

For the numerical values used in [11]

$$A_1 = 40 \text{ kgm sec}^2, \quad J_1 = 0.4 \text{ kgm sec}^2, \quad \omega_{i0} = 0$$

$$A_2 = A_3 = 850 \text{ kgm sec}^2, \; J_2 = J_3 = 8.5 \text{ kgm sec}^2, \; p_{i0} = 0.1 \text{ degree/sec}$$

$$(i = 1, 2, 3)$$

the intermediate processes at different values of are shown in Figs. 2 and 3. All the approximations satisfy the condition

$$p_i(\infty) = 0$$

When calculating the multipliers $\lambda_i + \varphi_i$ $(i = 1, 2, 3)$ the quantities λ_i $(i = 1, 2, 3)$ were neglected being of second order of smallness.

4. Isodromic control. An important application in the case when the rigid body is acted upon by a constant perturbing moment whose X_1, X_2, X_3 components are ϵ_1, ϵ_2, ϵ_3, respectively.

Let us state the problem; find the equation of an optimum control having an isodermic effect, which means that the action of the perturbing moment is compensated only by deviations of the controlling device. These deviations are determined from Formulas

$$U_i^* = -\varepsilon_i^* \qquad (i = 1, 2, 3) \qquad (4.1)$$

Consequently, the equations of motion of the body, when the perturbing moments are taken into account will have the same form as (1.5) if U_i is implied as

$$U_i - U_i^* = u_i \qquad (i = 1, 2, 3) \qquad (4.2)$$

With respect to these equations the problem of analytic design is formulated in the usual way if in the functional (1.11) the values U_i imply the difference (4.2). Its solution has obviously the form

$$u_i = -\rho p_i \qquad (i = 1, 2, 3) \qquad (4.3)$$

Similarly to 12 and 13 the control equations are combined, that is they contain both deviation signal and loading signal.

5. Optimum stabilization in a finite time interval. We shall consider the problem of analytic design, estimating the degree of stabilization of a rigid body from the functional

$$I(u) = \int_0^T e^{\delta t} W dt + k\omega^2(T), \qquad \omega^2(T) = \sum_{i=1}^3 p_i^2(T) \qquad (5.1)$$

Here k and δ are nonnegative constants. The multiplier $e^{\delta t}$ helps to speed up the damping of the intermediate processes.

If we assume now that the generating function is

$$\Psi(p(t), t) = \min_U \left(\int_t^T e^{\delta t} W dt + k\omega^2(T)\right) \quad (5.2)$$

we obtain the following Bellman equation

$$-\frac{\partial\Psi}{\partial t} = e^{\delta t} a \sum_{i=1}^{3} p_i^2 + A_1^{-1}(z_2p_3 - z_3p_2)\frac{\partial\Psi}{\partial p_1} +$$

$$+ A_2^{-1}(z_3p_1 - z_1p_3)\frac{\partial\Psi}{\partial p_2} + A_3^{-1}(z_1p_2 - z_2p_1)\frac{\partial\Psi}{\partial p_3} - \frac{1}{4}\sum_{i=1}^{3}\left(A_i^{-1}\frac{\partial\Psi}{\partial p_i}\right)^2 \quad (5.3)$$

$$\Psi(p(T), T) = k\omega^2(T)$$

We shall show one particular solution of the problem. Let

$$A_1 = A_2 = A_3 = A_1$$

Then Equation (5.3) can be satisfied by setting

$$\Psi = \rho(t) e^{\delta t} A_1 \sum_{i=1}^{3} p_i{}^2 \qquad (i = 1, 2, 3) \quad (5.4)$$

Here ψ is a smooth function satisfying the Riccati equation

$$\rho^{\cdot} = A_1^{-1}(\rho^2 - a) - \delta\rho \quad (5.5)$$

and assuming only positive values such that

$$A_1 e^{\delta T}\rho(T) = k, \qquad \rho(T) = kA_1^{-1}e^{-\delta T}$$

We shall investigate its solution setting $t = T - \tau$. We have

$$\rho_{2,1} = {}^1\!/_2 A_1\delta \pm \sqrt{({}^1\!/_2 A_1\delta)^2 + a} \quad (5.6)$$

The equation has the following solution

$$\rho = \frac{\rho_2(k^* - \rho_1) - \rho_1(k^* - \rho_2)\exp((\rho_2 - \rho_1)/A_1)\tau}{k^* - \rho_1 - (k^* - \rho_2)\exp((\rho_2 - \rho_1)/A_1)\tau} \quad (5.7)$$

This solution corresponds to the case when the constants $k > \rho_2$, $T > 0$ are such that the denominator of the fraction is positive.

The control U_1 is determined from Formulas (2.7) and has the form

$$U_i = -\rho(t) p_i \quad (5.8)$$

where $\rho(t)$ is the solution (5.7).

Let us consider the case when $\delta = 0$. Equation (5.3) can be satisfied by taking

$$\Psi = \rho(t) A_1 (p_1^2 + p_2^2 + p_3^2) \quad (5.9)$$

Here $\rho(t)$ is a smooth function satisfying the Riccati equation

$$\rho^{\cdot} = A_1^{-1}(\rho^2 - a) \quad (5.10)$$

The solution of Equation (5.10) is found from Formula (5.7) where ρ_2, ρ_1 are determined from the conditions $\rho^2 - a = 0$, $\rho = \pm\sqrt{a}$. The control U_1 has the form (5.8) where ρ is the solution (5.7), in which ρ_2, ρ_1 are determined from Formula (5.6).

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